Spinning jets

J. Eggers[1] and M. P. Brenner[2]
[1] *Universität Gesamthochschule Essen, Fachbereich Physik, 45117 Essen, Germany*
[2] *Department of Mathematics, MIT, Cambridge, MA 02139*

A fluid jet with a finite angular velocity is subject to centripetal forces in addition to surface tension forces. At fixed angular momentum, centripetal forces become large when the radius of the jet goes to zero. We study the possible importance of this observation for the pinching of a jet within a slender jet model. A linear stability analysis shows the model to break down at low viscosities. Numerical simulations indicate that angular momentum is expelled from the pinch region so fast that it becomes asymptotically irrelevant in the limit of the neck radius going to zero.

I. INTRODUCTION

A fluid jet emanating from a nozzle will become unstable and break up due to surface tension. Some 30 years ago, a series of papers [1–4] investigated the modifications Rayleigh's classical analysis would undergo if the jet performed a solid-body rotation. Such a rotation is easily imparted by spinning the nozzle at the appropriate frequency. The somewhat surprising result of the linear analysis is that the rotation always *destabilizes* the jet, a wavenumber k being unstable if

$$0 < kr_0 < (1 + L^{-1})^{1/2}, \tag{1}$$

with

$$L = \gamma/(\rho\Omega^2 r_0^3). \tag{2}$$

Here γ is the surface tension, ρ the density, Ω the angular frequency, and r_0 the unperturbed jet radius. Note that Ω appears in the denominator, so no rotation corresponds to $L \to \infty$, for which the stability boundary $0 < kr_0 < 1$ found by Plateau is recovered. The theoretical growth rates were found to be in reasonable agreement with experiment [4] and growth of disturbances for kr_0 larger than 1 was confirmed.

Recently it was pointed out by Nagel [5] that rotation might have an even more dramatic effect for the highly nonlinear motion near the point where the neck radius goes to zero. Assume for the sake of the argument that a cylinder of fluid of length w pinches uniformly, i.e. it retains its cylindrical shape. Then the total angular momentum is

$$M = \frac{\pi}{2}\rho\Omega r_0^4 w, \tag{3}$$

and the volume $V = \pi r_0^2 w$ is constrained to remain constant as r_0 goes to zero. The total interface pressure corresponding to the outward centripetal force is found to be $p_c = \rho r_0^2\Omega^2/2$, and thus

$$p_c = 2M^2/(V^2\rho r_0^2). \tag{4}$$

As r_0 goes to zero, this outward pressure will *dominate* the surface tension pressure γ/r_0, raising the possibility that rotation is a singular perturbation for pinching: an arbitrarily small amount of angular momentum, caused by a symmetry breaking, could modify the breaking of a jet.

However, a jet does not pinch uniformly, but rather in a highly localized fashion [6]. If the above argument is applied to pinching, it must correspond to a rapidly spinning thread of fluid surrounded by almost stationary fluid. Frictional forces represented by the viscosity of the fluid will lead to a diffusive transport of angular momentum out of the pinch region, thus reducing its effect. Determining which effect dominates requires a fully nonlinear calculation, including effects of surface tension, viscosity, inertia, and centripetal forces. In the spirit of earlier work for $\Omega = 0$ [7,8] we will derive a one dimensional model, which only takes into account the leading order dependence of the velocity field on the radial variable. This will be done in the next section, together with a comparison of the linear growth rates between the model and the full Navier-Stokes calculation. In the third section we analyze the nonlinear behavior. First we investigate possible scaling solutions of the model equations, then we compare with numerical simulations. In the final section, we present some tentative conclusions.

II. THE MODEL

In our derivation of the slender jet model we closely follow [7]. The Navier-Stokes equation for an incompressible fluid of viscosity ν read in cylindrical coordinates:

$$\partial_t v_r + v_r \partial_r v_r + v_z \partial_z v_r - v_\phi^2/r = -\partial_r p/\rho + \nu(\partial_r^2 v_r + \partial_z^2 v_r + \partial_r v_r/r - v_r/r^2),$$

$$\partial_t v_z + v_r \partial_r v_z + v_z \partial_z v_z = -\partial_z p/\rho + \nu(\partial_r^2 v_z + \partial_z^2 v_z + \partial_r v_z/r),$$

$$\partial_t v_\phi + v_r \partial_r v_\phi + v_z \partial_z v_\phi + v_r v_\phi/r = \nu(\partial_r^2 v_\phi + \partial_z^2 v_\phi + \partial_r v_\phi/r - v_\phi/r^2),$$

with the incompressibility condition

$$\partial_r v_r + \partial_z v_z + v_r/r = 0.$$

Here we have assumed that the velocity field does not depend on the angle ϕ. Exploiting incompressibility, v_z and v_r can be expanded in a power series in r:

$$\begin{aligned} v_z(z,r) &= v_0(z) + v_2(z) r^2 + \ldots \\ v_r(z,r) &= -\frac{v_0'(z)}{2} r - \frac{v_2'(z)}{4} r^3 - \ldots . \end{aligned} \tag{5}$$

Here and in the following a prime refers to differentiation with respect to z.

The crucial trick to make an expansion in r work in the presence of rotation is to rewrite $v_\phi(z,r)$ in terms of the angular momentum per unit length $\ell(z)$ of the corresponding solid body rotation:

$$v_\phi(z,r) = \frac{2\ell(z)}{\pi\rho h^4(z)} r + b r^3 + \ldots . \tag{6}$$

Here $h(z)$ is the local thread radius, hence no overturning of the profile is allowed. Just as without rotation, the equation of motion for $h(z,t)$ follows from mass conservation based on the leading order expression for v_z:

$$\partial_t h + v_0 h' = -v_0' h/2 \quad . \tag{7}$$

Finally, the pressure is expanded according to

$$p(z,r) = p_0(z) + p_2(z) r^2 + \ldots . \tag{8}$$

Plugging this into the equation of motion for v_r, to leading order in r one finds the balance

$$p_2 = \frac{2\ell^2}{\pi^2 \rho h^8}, \tag{9}$$

while the leading balance for the v_z-equation remains

$$\partial_t v_0 + v_0 v_0' = -p_0'/\rho + \nu(4v_2 + v_0''). \tag{10}$$

Lastly, the v_ϕ-equation leads to

$$\partial_t \ell + \ell v_0' + 4\ell v_0 h'/h + v_0 h^4 (\ell/h^4)' = \nu h^4 (4\pi\rho b + (\ell/h^4)'') \tag{11}$$

to leading order.

Equations (9)-(11) contain the unknown functions p_0, v_2, and b which need to be determined from the boundary conditions. The normal stress balance $\mathbf{n}\sigma\mathbf{n} = \gamma\kappa$ gives

$$p_0 + p_2 h^2 = \gamma\kappa - v_0',$$

where κ is the sum of the principal curvatures. As in the case without rotation, the tangential stress balance $\mathbf{n}\sigma\mathbf{t} = 0$ gives

$$-3v_0'h' - v_0''h/2 + 2v_2h = 0$$

for $\mathbf{t}$ pointing in the axial direction, but a new condition

$$\pi\rho hb = h'(\ell/h^4)'$$

for $\mathbf{t}$ pointing in the azimuthal direction. Putting this together, one is left with a closed equation for h, v_0, and ℓ:

$$\begin{aligned} \partial_t h + vh' &= -v'h/2 \\ \partial_t v + vv' &= -\frac{\gamma}{\rho}\kappa' + \frac{2}{\rho^2\pi^2}(\ell^2/h^6)' + 3\nu(v'h^2)'/h^2 \\ \partial_t l + (vl)' &= \nu(h^4(\ell/h^4)')', \end{aligned} \tag{12}$$

where we have dropped the subscript on v_0. The same equations were derived independently by Horvath and Huber [10].

The most obvious way to test this model is to compare with the known results for the stability of the full Navier-Stokes equation. To that end we linearize (12) about a solution with constant radius r_0 and rotation rate Ω:

$$\begin{aligned} h(z,t) &= r_0(1 + \epsilon e^{\omega t}\cos(kz)) \\ v(z,t) &= -2\epsilon\frac{\omega}{k}e^{\omega t}\sin(kz) \\ \ell(z,t) &= \frac{\pi}{2}\rho\Omega r_0^4(1 + \epsilon e^{\omega t}\alpha\cos(kz)). \end{aligned} \tag{13}$$

Eliminating α, this leads to the equation

$$\bar{\omega}^3 + \frac{4\bar{k}^2}{Re}\bar{\omega}^2 + \frac{\bar{k}^2}{2}(-1 + \bar{k}^2 + L^{-1} + 6\bar{k}^2/Re^2)\bar{\omega} + \frac{\bar{k}^4}{2Re}(-1 + \bar{k}^2 - L^{-1}) = 0, \tag{14}$$

where $\bar{k} = kr_0$ and $\bar{\omega} = \omega(\rho r_0^3)^{1/2}/\gamma^{1/2}$ are dimensionless. We have introduced the Reynolds number $Re = (\gamma r_0)^{1/2}/(\rho^{1/2}\nu)$, based on a balance of capillary and viscous forces. Note that this convention differs from that of [4]. Putting $\bar{\omega} = 0$ one reproduces the exact stability boundaries (1). However one can see that the inviscid limit $Re \to \infty$ is a very singular one, in disagreement with the full solution. Namely, for this limit one finds the three branches

$$\bar{\omega}_{1/2}^2 = \frac{\bar{k}^2}{2}(1 - \bar{k}^2 - L^{-1}), \quad \bar{\omega}_3 = Re^{-1}\frac{\bar{k}^2(1 - \bar{k}^2 + L^{-1})}{\bar{k}^2 - 1 + L^{-1}}. \tag{15}$$

Thus $\bar{\omega}_3$ is the only unstable mode in the range $1 - L^{-1} < \bar{k}^2 < 1 + L^{-1}$, but goes to zero when the viscosity becomes small. The reason for this behavior, which is not found in the solution of the full equations, lies in the appearance of a very thin boundary layer for small viscosities [3]. Namely, Rayleigh's stability criterion for a rotating fluid implies that the interior of the fluid is *stabilized*. This forces any disturbance to be confined to a boundary layer of thickness

$$\delta = \frac{\omega}{2\Omega k}$$

near the surface of the jet, and δ becomes very small for $\bar{k} \approx 1$. But this additional length scale is not captured by our slender jet expansion. Only for high viscosities is the boundary layer smoothed out sufficiently, and from (14) one finds the dispersion relation

$$\bar{\omega} = \frac{Re}{6}(1 - \bar{k}^2 + L^{-1}), \tag{16}$$

which is consistent with the full theory in the limit of small $\bar{k}$.

III. NONLINEAR BEHAVIOR

Our main interest lies of course in the behavior close to pinch-off. Close to the singularity, one expects the motion to become independent of initial conditions, so it is natural to write the equations of motion in units of the material parameters of the fluid alone. In addition to the known [9] units of length and time, $\ell_\nu = \nu^2\rho/\gamma$ and $t_\nu = \nu^3\rho^2/\gamma^2$, one finds an angular momentum scale $\ell_0 = \nu^5\rho^2/\gamma^2$. Note that this scale is only about $1.9 \cdot 10^{-14}$ g cm/s for water, corresponding to a frequency of $2 \cdot 10^{-11} s^{-1}$ for a 1 mm jet, so even the smallest amount of rotation will be potentially relevant. Rewriting the equations of motion (12) in units of ℓ_ν, t_ν, and ℓ_0, one can effectively put $\rho = \nu = \gamma = 1$, leading to a universal form of the equations, independent of the type of fluid.

In addition, one can look for similarity solutions [11] of the form

$$\begin{aligned} h(z,t) &= t'^{\alpha_1}\phi(z'/t'^{\beta}) \\ v(z,t) &= t'^{\alpha_2}\psi(z'/t'^{\beta}) \\ \ell(z,t) &= t'^{\alpha_3}\chi(z'/t'^{\beta}), \end{aligned} \tag{17}$$

where $t' = t_0 - t$ and $z' = z - z_0$ are the temporal and spatial distances from the singularity where h goes to zero. We have assumed that everything has been written in units of the natural scales ℓ_ν, t_ν, and ℓ_0. By plugging (17) into the equations of motion, and looking for solutions that balance the t'-dependence, one finds a unique set of values for the exponents:

$$\alpha_1 = 1, \quad \alpha_2 = -1/2, \quad \alpha_3 = 5/2, \quad \beta = 1/2.$$

In addition, one obtains a set of three ordinary differential equations for the similarity functions ϕ, ψ, and χ. So far we have not been able to find consistent solutions to these equations, which match onto a solution which is static on the time scale t' of the singular motion. This is a necessary requirement since the fluid will not be able to follow the singular motion as one moves away from the pinch point.

This negative result is consistent with simulations of the equations for a variety of initial conditions. To avoid spurious boundary effects, we considered a solution of (12) with periodic boundary conditions in the interval $[-1, 1]$ and an additional symmetry around the origin. This ensures that the total angular momentum is conserved exactly. We took the fluid to be initially at rest and the surface to be

$$h_{init}(z) = r_0(1 + 0.3\cos(2\pi x)), \tag{18}$$

with $r_0 = 0.1$. The angular momentum was distributed uniformly with the initial value ℓ_{init}, corresponding to

$$L = \frac{\pi^2}{4}\frac{\gamma\rho r_0^5}{\ell_{init}^2}.$$

Figures 1 and 2 show a numerical simulation of (12) with $Re = 4.5$ and $L = 0.25$ using a numerical code very similar to the one described in [12]. Written in units of the intrinsic angular momentum scale, $\ell_{init}/\ell_0 = 6 \cdot 10^3$, so ℓ is potentially relevant. The thread pinches on either side of the minimum, pushing fluid into the center. As seen in the profiles of ℓ, the angular momentum is expelled very rapidly from the regions where h goes to zero and also concentrates in the center. This is confirmed by a plot of the minimum of ℓ versus the minimum of h. On the basis of the similarity solution (17), a power law $\ell_{min} \propto h_{min}^{5/2}$ is to be expected. Instead, Fig. 3 shows that ℓ_{min} decays more rapidly, making angular momentum asymptotically irrelevant. Indeed, a comparison of the similarity function ϕ as found from the present simulation shows perfect agreement with the scaling theory in the absence of rotation [11]. The behavior of ℓ_{min} should in principal be derivable from the linear equation for ℓ with known time-dependent functions $h(z,t)$ and $v(z,t)$. Unfortunately, ℓ_{min} does not seem to follow a simple scaling law except below $h = 3 \cdot 10^{-4}$, where the power is close to 3.13. It is not clear how to extract this power analytically.

One might think that by increasing the angular momentum the system would cross over to a different behavior. To test this, the initial angular momentum was doubled to give $L = 0.0625$. At $L = 0.5$ centripetal and surface tension forces are balanced, so decreasing L significantly below this value will cause rotation to be important initially. Indeed, instead of pinching down immediately, the fluid is first pulled into a narrow disc, while the radius of the surrounding fluid remains constant, cf. Fig. 4. Eventually this outward motion stops, as surface tension and centripetal forces reach an equilibrium. Only then does the fluid pinch down at the edge of the disc. The behavior close to the pinch point is however exactly the same as for smaller angular momentum. As a word of caution, one must add that our model is certainly not valid at the edges of the disk, where slopes become very large. In fact, the very sharp gradients encountered in this region may be due to the fact that the fluid really wants to develop *plumes*. As is observed for low viscosity [12], the viscous terms prevent the interface from overturning, but at the cost of producing unrealistically sharp gradients.

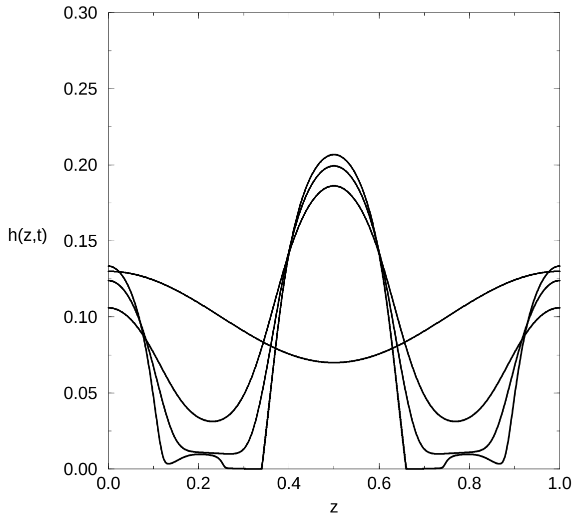

FIG. 1. The height profile in a numerical simulation with $Re = 4.5$ and $L = 0.25$. Shown are the initial condition, and the times when the minimum has reached $h_{min} = 10^{-1.5}, 10^{-2}$, and 10^{-5}, at the end of the computation.

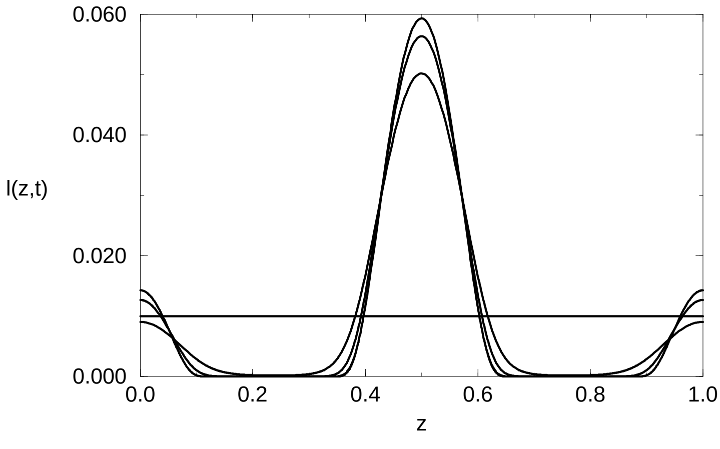

FIG. 2. The angular momentum profiles $\ell(z, t)$ corresponding to Fig. 1.

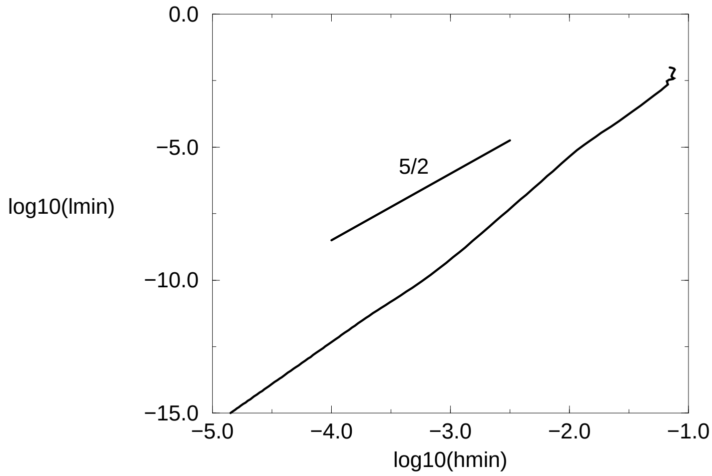

FIG. 3. The minimum value of the angular momentum as function of the minimum height. It is found that ℓ_{min} decreases faster than $h_{min}^{5/2}$, which would exactly balance surface tension and centripetal forces.

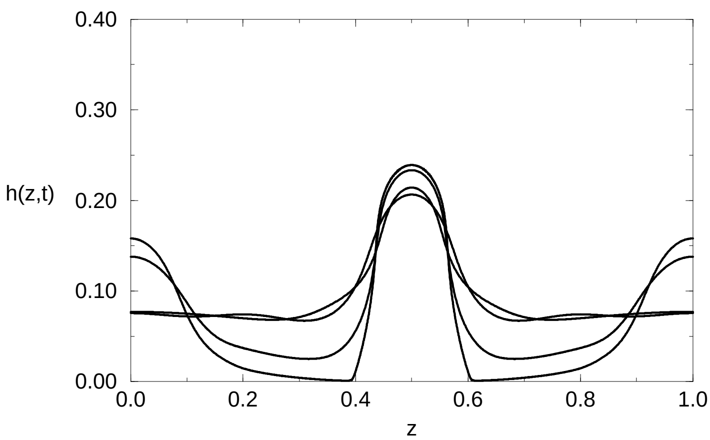

FIG. 4. A numerical simulation with twice the angular momentum of Fig. 1. The height profiles are shown in time intervals of 0.05 and at the end of the simulation. Centripetal forces draw the fluid out into a disc.

IV. CONCLUSIONS

The present investigation is only a first step towards the understanding of the role of rotation in droplet pinching. A major challenge lies in finding a description valid at low viscosities. This can perhaps be done by incorporating the boundary layer structure near the surface into the slender jet approximation. The relevance of this lies in the fact that angular momentum is potentially more important at low viscosities, when there is less frictional transport out of the pinch region. In fact it can be shown that the inviscid version of (12) does not describe breakup at all, since centripetal forces will always dominate asymptotically. This result is of course only of limited use since the model equations are definitely flawed in that regime.

In addition, there remains the possibility that a region in parameter space exists where angular momentum modifies breakup even at finite viscosities. We cannot make a definite statement since the additional variable makes it hard to scan parameter space completely. Finally, spinning jets have not received much attention in terms of experiments probing the non-linear regime. The discs found at high spinning rates (cf. Fig. 4) are a tantalizing new feature, and to our knowledge have not been found experimentally. The lowest value of L reported in [4] is 0.43, which is even larger than the value of Fig. 1. However, 0.0625 would easily be reachable by increasing the jet radius.

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